ASTROPHYSICS Yr 2 2011-2012

Session 4 – Orbits & the Solar System 2

Using Kepler's Laws

$\mathbf{r} = \frac{a(1-e^2)}{1+e\cos\theta}$ **Polar form of ellipse equation;**

Not necessary to prove it but used widely.

r = radius vector

e = eccentricity

a = semi major axis

θ = true anomaly.

$\theta = 0^o \rightarrow$ perihelion distance; $r_1 = a(1 - e)$

$\theta = 180^o \rightarrow$ aphelion distance $r_2 = a(1 + e)$

If a is in A.U.'s & orbital period P is in sidereal years; Kepler's 3rd law →

$\mathbf{A^3 = P^2}$.

Examples (using the data in Appendix 1 at the end of Session 3)

a. The Earth's perihelion distance;

$\mathbf{r_1 = a(1 - e)}$

$= 1.496 \times 10^{11} (1 - 0.017) = 1.47 \times 10^{11}$ metres; i.e. ~ 150 million km.

b. The semi-major axis of Mars 'orbit is given in A.U.'s by:

$a^3 = (1.881)^2$; (1.881 being the orbital period of Mars in sidereal years).

i.**e. a = 1.52 A.U. or about 227 million km.**

c. Mars' distance from the Sun when the true anomaly equals 60°.

The polar equation for the ellipse tells us that:

$$r = \frac{1.52(1-0.0934^2)}{1+(0.0934\ \cos 60)} = \mathbf{1.4395\ A.U.}$$

Newton's Law of Gravitation

Newton's Law of gravity gives a deeper insight into Kepler's Laws as well as being able to prove them. All we need here is the inverse square law itself together with the laws of conservation of energy and angular momentum.

For any two bodies of mass 'M' and 'm', separated by a distance 'r', the attractive force 'F' between them is given by:

$\mathbf{F} = \frac{GMm}{r^2}$; G is the Universal Constant of Gravitation; equal to **$6.673 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$**.

Proving Kepler's Laws

The First Law

This is actually by far the most difficult one to prove even with the use of calculus (Newton in his Principia gave a geometrical proof which, so I've heard is notoriously difficult to follow). It is not needed for this course so we'll leave it at that – keen ones among you will readily find an explanation of the proof on the Internet.

The Second Law

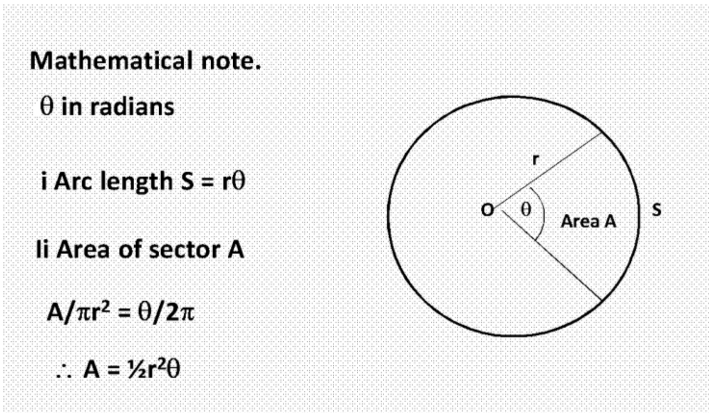
This simply uses the law of conservation of angular momentum 'L'. For a planet orbiting at distance r from the Sun, L is given by:

$L = mrv = mr^2\omega$;

m = mass of planet; v = planet's orbital speed.

Area 'A' swept out by the radius vector (θ in radians) is given by:

A = ½ r $^2\theta$: *provided θ is small; i.e. r can be regarded as constant. In this case A = area of equivalent sector of circle of radius r.*



Mathematical note.

θ in radians

i Arc length S = rθ

Ii Area of sector A

$A/\pi r^2 = \theta/2\pi$

$\therefore$ A = ½r$^2\theta$

So rate at which the area is being swept out is

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega = \frac{L}{2m}$$

Note: We're not actually differentiating anything here; dA/dt simply means the rate at which A is changing with time; likewise for dθ/dt which equals the angular velocityω.

The conservation of angular momentum means that L is constant around the orbit (as of course is m) and so $\frac{dA}{dt}$ is constant thus proving Kepler's Second Law.

The Third Law – (Definitely not necessary to be able to do this for the exam).

Here we assume that both the total energy 'E' of the planet and its angular momentum 'L' are conserved; i.e. they are constant around the orbit. We then consider their values at perihelion and at aphelion where the radius vectors and orbital speeds are r_1, v_1 and r_2, v_2 respectively. Finally we use Newton's Law of Gravitation to tell us that the potential energy of the planet when at distance r from the Sun is given by:

P.E. = $-\frac{GMm}{r}$;

M = mass of Sun;

(P.E. = 0 when r = ∞ and becomes increasingly negative for smaller values of r)

Now take a deep breath and plod on!

Total energy (kinetic plus potential) at perihelion and aphelion:

$$E = ½mv_1^2 - \frac{GMm}{r_1} = ½mv_2^2 - \frac{GMm}{r_2}$$

$$\rightarrow ½mv_1^2 - ½mv_2^2 = \frac{GMm}{r_1} - \frac{GMm}{r_2}$$

Eliminating m;

$$½(v_1^2 - v_2^2) = GM(\frac{1}{r_1} - \frac{1}{r_2})$$

Bringing in the angular momentum L;

$$v_1 = \frac{L}{mr_1} \text{ and } v_2 = \frac{L}{mr_2}$$

So;

$$\frac{L^2}{2m^2}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) = GM(\frac{1}{r_1} - \frac{1}{r_2})$$

$$\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) = (\frac{1}{r_1} + \frac{1}{r_2})(\frac{1}{r_1} - \frac{1}{r_2})$$

Thus;

$$\frac{L^2}{2m^2} = \frac{GM}{(\frac{1}{r_1} + \frac{1}{r_2})}$$

In proving the second Law we showed that the rate at which the area swept out by the radius vector is;

$$\frac{dA}{dt} = \frac{L}{2m}$$

The total area of the orbit (remember from the properties of the ellipse, this is equal to πab) is swept out in a time $\frac{\pi ab}{\frac{dA}{dt}}$ which equals the orbital period P so;

$$P = \frac{\pi ab}{\frac{L}{2m}} \text{ and thus } P^2 = \frac{(\pi ab)^2}{\frac{L^2}{4m^2}} = \frac{2(\pi ab)^2}{\frac{L^2}{2m^2}} = \frac{2(\pi ab)^2}{GM}(\frac{1}{r_1} + \frac{1}{r_2})$$

Last bit!

From the properties of the ellipse;

$b^2 = a^2 (1 - e^2)$

$r_1 = a(1 - e)$

$r_2 = a(1 + e)$

So $b^2 = r_1 r_2$ and $a = \frac{r_1+r_2}{2}$

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1+r_2}{r_1 r_2} = \frac{2a}{b^2}$$

So;

$$P^2 = \frac{2(\pi ab)^2}{GM} \times \frac{2a}{b^2}$$

Which finally gives us:

$$\mathbf{P^2 = \frac{4\pi^2 a^3}{GM}}$$

Lo and beholdd! Kepler's third Law.

The quantity GM is used frequently and it is given the symbol 'μ'

For the Sun M = 1.989 × 10^{30}kg

And **$\mu = 1.327 \times 10^{20}\ \mathrm{m^3 s^{-1}}$**

The Speed of an Orbiting Planet

One of the most important numbers that a rocket scientist needs to know is the speed of an object (planet, interplanetary space probe etc.) moving on an elliptical orbit; not surprisingly, astronomers are quite interested in this too. For circular orbits there is a fairly simple equation for the orbital speed which you may already know and which we'll come to shortly anyway. For an elliptical orbit things are a little bit more involved as we shall now see.

We begin by finding an alternative expression for the total energy E of an object on an elliptical orbit.

This time we add together the kinetic and potential energy for the planet at perihelion and at aphelion (using the same expressions for these values as in the previous section).

$$\tfrac{1}{2}\mathrm{m}v_1^2 - \frac{GMm}{r_1} + \tfrac{1}{2}\mathrm{m}v_2^2 - \frac{GMm}{r_2} = 2\mathrm{E}$$

Again, using the relations; $V_1 = \frac{L}{mr_1}$ etc.; we have

$$\frac{L^2}{2m^2}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) - \mathrm{GMm}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = 2E$$

Also from the previous section;

$$\frac{L^2}{2m^2} = \frac{\mathrm{GM}}{(\frac{1}{r_1}+\frac{1}{r_2})}$$

And in addition;

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} = \left(\frac{1}{r_1} + \frac{1}{r_2}\right)^2 - \frac{2}{r_1 r_2}$$

So;

$$2\mathrm{E} = \mathrm{GMm}\left\{\frac{\left(\frac{1}{r_1}+\frac{1}{r_2}\right)^2 - \frac{2}{r_1 r_2}}{\left(\frac{1}{r_1}+\frac{1}{r_2}\right)} - \left(\frac{1}{r_1} + \frac{1}{r_2}\right)\right\}$$

Ignoring your headache;

$$2\mathrm{E} = \text{-}\ \mathrm{GMm}\left\{\frac{\frac{2}{r_1 r_2}}{\left(\frac{1}{r_1}+\frac{1}{r_2}\right)}\right\}$$

$$\text{So } \mathrm{E} = \frac{-GMm}{2}\left\{\frac{\frac{2}{r_1 r_2}}{\frac{r_1+r_2}{r_1 r_2}}\right\} = \frac{-GMm}{r_1+r_2}; \quad r_1 + r_2 = 2a$$

And finally $\mathrm{E} = \frac{-GMm}{2a}$

All in all pretty tedious stuff but at least the equation for the total energy in terms of the orbit's semi-major axis is a simple one, so remember it!

This simple formula will now give us the (variable) velocity of a body moving on an elliptical orbit.

One more time, the total energy of the body can now be expressed as:

$\frac{-GMm}{2a} = ½\mathrm{m}v^2 - \frac{GMm}{r}$; or getting rid of m;

$$\frac{-GM}{2a} = ½v^2 - \frac{GM}{r}$$

This is known as Newton's 'vis viva' or 'living force' equation (seems a bit spooky coming from someone like Newton).

Then;

$$v^2 = 2\mathrm{GM}\left\{\frac{1}{r} - \frac{1}{2a}\right\}$$

$$v^2 = GM\{\frac{2}{r} - \frac{1}{a}\}$$

$$\boldsymbol{v} = \sqrt{\boldsymbol{GM}\{\frac{\boldsymbol{2}}{\boldsymbol{r}} - \frac{\boldsymbol{1}}{\boldsymbol{a}}\}}$$

Circular Orbits

For a circular orbit, r ≡ a and is constant of course; the total energy of the orbiting body is then;

$$\mathrm{E} = \frac{-GMm}{2r}$$

And the orbital velocity is given by;

$$\mathrm{V} = \sqrt{\frac{GM}{r}}$$

A perhaps more familiar approach to the velocity for a circular orbit is to begin with the force on the orbiting body;

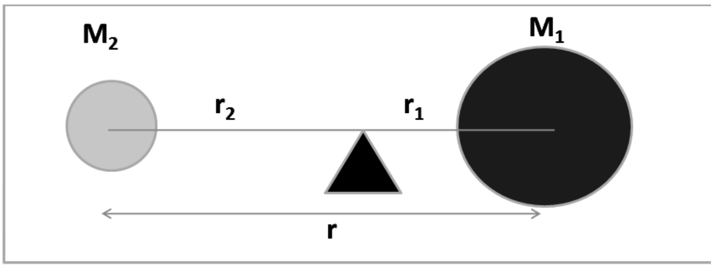
$$\frac{GMm}{r^2} = \frac{mv^2}{r};$$

Giving;

$$\mathbf{v} = \sqrt{\frac{GM}{r}}$$

as before.

Newton's version of Kepler's 3rd law (for circular orbits)



Two bodies will orbit their common centre of mass

or

Barycentre.

Force between the two bodies:

$$\boldsymbol{F} = \frac{\boldsymbol{GM_1M_2}}{\boldsymbol{r^2}};$$

Also for circular orbits:

$$\boldsymbol{F} = \frac{\boldsymbol{M_1v_1^2}}{\boldsymbol{r_1}} = \frac{\boldsymbol{M_2v_2^2}}{\boldsymbol{r_2}}$$

For orbital period P:

$$\boldsymbol{P = \frac{2\pi r_1}{v_1} = \frac{2\pi r_2}{v_2}}$$

i.e.:

$$\boldsymbol{v_1^2 = \frac{4\pi^2 r_1^2}{P^2} \; and \; v_2^2 = \frac{4\pi^2 r_2^2}{P^2}}$$

Inserting these into the equations for **F**:

$$\boldsymbol{\frac{GM_1M_2}{r^2} = \frac{4\pi^2 M_1 r_1}{P^2} \rightarrow \frac{GM_2}{r^2} = \frac{4\pi^2 r_1}{P^2}}$$

And:

$$\boldsymbol{\frac{GM_1M_2}{r^2} = \frac{4\pi^2 M_2 r_2}{P^2} \rightarrow \frac{GM_1}{r^2} = \frac{4\pi^2 r_2}{P^2}}$$

Add these two together:

$\boldsymbol{\frac{G(M_1+M_2)}{r^3} = \frac{4\pi^2}{P^2}};$ Noting that $r = r_1 + r_2$.

This is Newton's version of Kepler's 3rd law of planetary motion.

As we'll see later, this provides the only means of directly measuring the mass of a star.

Finally – for rocket scientists!

Escape Velocity

The total work which needs to be done by an object of mass m against the gravitational force in order to escape from the surface of a body of mass M and radius r is equal to;

$$\mathrm{E} = \int_r^\infty \frac{-GMm}{r^2} dr = \frac{GMm}{r};$$

This work; i.e. energy is provided by the object's kinetic energy so;

$\tfrac{1}{2}\mathrm{mv}^2 = \frac{GMm}{r}$ and so;

$\mathrm{v} = \sqrt{\frac{2GM}{r}}$ *The escape velocity*

Example

For the Moon; $\mathrm{M} = 7.349 \times 10^{22}$ kg

$r = 1.738 \times 10^6$m

So v = $\sqrt{\frac{2\times6.673\times10^{-11}\times7.349\times10^{22}}{1.738\times10^{6}}}$

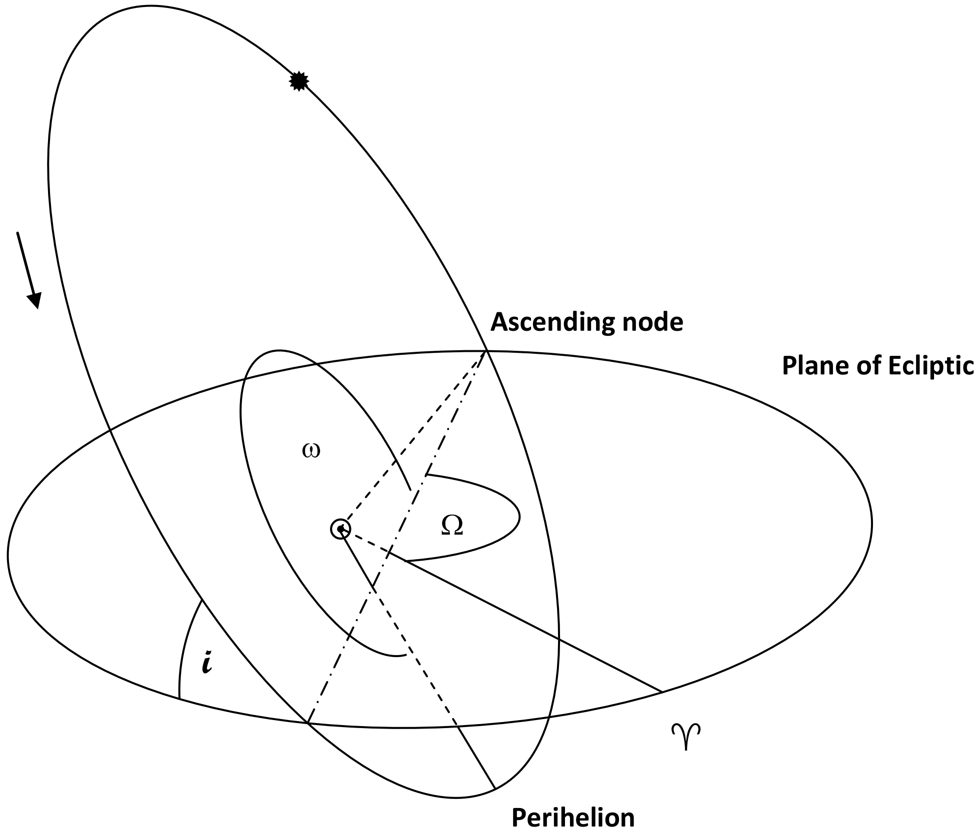
= 2.376 km/sec; i.e. about 1½ miles per second, compared to the Earth's 7 miles per second.

Orbits in Three Dimensions *(for interest only)*

The importance of ecliptic coordinates (Session 3) lies in dealing with the orbits of comets and many (possibly rogue) asteroids which have high orbital inclinations. This means we have to consider orbits in three dimensions. The orbit of say a newly discovered comet will initially be seen projected against the celestial sphere; i.e. the background stars. Careful observations then enable the true orbit to be 'framed' and oriented correctly within the three dimensional space surrounding the Solar System. A series of key parameters 'fixes' the shape and orientation of the orbit. These are called the *orbital elements* of the comet, asteroid etc., three of which are already familiar.

The Orbital Elements are:

- The semi-major axis 'a'.
- The eccentricity 'e'.
- The true anomaly 'θ'.
- The inclination of the plane of the orbit to the plane of the ecliptic 'i'.
- The (ecliptic) longitude of the ascending node 'Ω'.
- The argument of perihelion 'ω'. This determines the tilt of the orbit's line of apsides relative to the plane of the ecliptic.



The first two of these as discussed previously determine the shape of the orbit and the true anomaly fixes the current position of the object on the orbit. The other three elements determine the orientation of the orbit with respect to the plane of the ecliptic and a reference direction which is usually the First Point of Aires. So not surprisingly, these three elements give the orbit's orientation in three dimensional space.

Prograde & Retrograde Orbits

If you could look down on the Sun's North Pole from some distance away, you'd see that all the major planets are orbiting in an anticlockwise direction. In addition almost all of the planet's moons would be doing the same. This orbital direction is called *prograde*. The odd moon; e.g. Saturn's outer moon Phoebe together with many comets, orbit in the opposite direction. Such an orbit is called a *retrograde orbit.*

Summary

Here listed together are the important equations.

$r = \frac{a(1-e^2)}{1+e\cos\theta}$	**Polar equation of the ellipse**
$r_1 = a(1-e)$	**Periapsis distance**
$r_2 = a(1+e)$	**Apasis distance**
πab	**Area of the ellipse**
$\frac{a^3}{P^2} = 1$	**Kepler's 3rd Law – for use only in the Solar System**
	a in A.U.'s; P in Earth sidereal years
$P^2 = \frac{4\pi^2 a^3}{GM}$	**Kepler's 3rd Law for general use**
$E = \frac{-GMm}{2a}$	**Total energy for an elliptical orbit**
$v = \sqrt{GM\{\frac{2}{r} - \frac{1}{a}\}}$	**Speed in an elliptical orbit**
$V = \sqrt{\frac{GM}{r}}$	**Speed in a circular orbit**

$$\frac{G(M_1+M_2)}{a^3} = \frac{4\pi^2}{P^2}$$

Newto**n's version of Kepler's 3rd law**

$$v = \sqrt{\frac{2GM}{r}}$$

Escape velocity