Frosty The Snowman

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1 Introduction

Frosty the snowman is made from two uniform spherical snowballs, of radii 2R and 3R. The smaller (which is his head) stands on top the larger. As each snowball melts, its volume decreases at a rate which is directly proportional to its surface area. The constant of proportionality being the same for each snowball. During melting, the snowballs remain spherical and uniform.

When frosty is half his initial height, show that the ratio of his volume to his initial volume is 37 : 224. Let V and h denote Frosty's total volume and height, respectively, at time t. Show that, for 2R < h ≤ 10R:

$$\frac{dV}{dh} = \frac{\pi}{8}(h^2 + 4R^2)$$

And derive the corresponding expression for 0 ≤ h < 2R. Sketch $\frac{dV}{dh}$ as a function of h, for 4R ≥ h ≥ 0, hence give a rough sketch of V as a function of h.

2 Initial Conditions

Given initial radii:

$$r_{1_0} = 2R$$
$$r_{2_0} = 3R$$
$$\therefore h_0 = 10R$$

Also that:

$$\frac{dV_1}{dt} = -kA_1$$
$$\frac{dV_2}{dt} = -kA_2$$

Where k is a constant with unit m/s, since A is the surface area of a sphere with units m^2 and $\frac{dV}{dt}$ is the rate of volume increase with units m^3/s.

3 Determination of Volume at Half the Initial Height

Since the initial radii differ by R, and both spheres have the same constant k determining the rate at which they melt, their radii will remain R apart as the snowballs melt until the smaller snowball has completely melted and $r_1 = 0$ (and h = 2R).

At half the initial height:

$$h = \frac{h_0}{2} = \frac{10R}{2} = 5R$$
$$h = 2(r_1 + r_2)$$
$$r_1 = r_2 - R$$

$$\therefore h = 2(r_2 - R + r_2)$$
$$h = 2(2r_2 - R)$$
$$5R = 2(2r_2 - R)$$
$$5R = 4r_2 - 2R$$
$$4r_2 = 7R$$
$$r_2 = 1.75R$$

$$\therefore r_1 = 0.75R$$

Using these values to calculate the volume:

$$V_{\frac{h}{2}} = \frac{4}{3}\pi(r_1^3 + r_2^3)$$
$$V_{\frac{h}{2}} = \frac{4}{3}\pi((0.75R)^3 + (1.75R)^3)$$
$$V_{\frac{h}{2}} = \frac{4}{3}\pi(0.421875R^3 + 5.359375R^3)$$
$$V_{\frac{h}{2}} = \frac{4}{3}\pi R^3 * 5.78125$$

And a quick calculation of the initial volume:

$$V_0 = \frac{4}{3}\pi(r_{1_0}^3 + r_{2_0}^3)$$
$$V_0 = \frac{4}{3}\pi((2R)^3 + (3R)^3)$$
$$V_0 = \frac{4}{3}\pi(8R^3 + 27R^3)$$
$$V_0 = \frac{4}{3}\pi R^3 * 35$$

Now compare the two volumes:

$$
\begin{aligned}
V_{\frac{h}{2}} &: V_0 \\
\frac{4}{3}\pi R^3 * 5.78125 &: \frac{4}{3}\pi R^3 * 35 \\
5.78125 &: 35
\end{aligned}
$$

Finally multiply both sides by 6.4 to get:

$$37 : 224$$

Therefore it has been shown that the half-height volume and initial volume are in the ratio $37 : 224$ ratio as desired.

4 Rate of Change of Volume With Respect to Height

Since the upper snowball melts before the lower snowball, the expression for the volume in terms of height is not a smooth function. While the curve is continuous, the first derivative of V with respect to h has a jump discontinuity at h = 2R. It must therefore be given in two parts, and these will be derived here.

Firstly for 2R < h ≤ 10R:

$$
\begin{aligned}
h &= 4r_1 + 2R \\
h - 2R &= 4r_1 \\
r_1 &= \frac{h - 2R}{4}
\end{aligned}
$$

$$
\begin{aligned}
h &= 4r_2 - 2R \\
h + 2R &= 4r_2 \\
r_2 &= \frac{h + 2R}{4}
\end{aligned}
$$

$$
\begin{aligned}
V &= \frac{4}{3}\pi(r_1^3 + r_2^3) \\
V &= \frac{4}{3}\pi((\frac{h - 2R}{4})^3 + (\frac{h + 2R}{4})^3) \\
V &= \frac{\pi}{3 * 16}((h - 2R)^3 + (h + 2R)^3) \\
V &= \frac{\pi}{3 * 16}(h^3 - 6h^2R + 12hR^2 - 8R^3 + h^3 + 6h^2R + 12hR^2 + 8R^3) \\
V &= \frac{\pi}{3 * 16}(2h^3 + 24hR^2) \\
V &= \frac{\pi}{3 * 8}(h^3 + 12hR^2)
\end{aligned}
$$

Now differentiating:

$$\frac{dV}{dh} = \frac{\pi}{3*8}(3h^2 + 12R^2)$$
$$\frac{dV}{dh} = \frac{\pi}{8}(h^2 + 4R^2) \quad : \quad 2R < h \leq 10R$$

Thus an expression for the rate of change of volume in terms of the height has been determined while there are two snowballs.

Secondly, an expression shall be derived for 0 ≤ h < 2R:

$$h = 2r_2$$
$$r_2 = \frac{h}{2}$$

$$V = \frac{4}{3}\pi r_2^3$$
$$V = \frac{4}{3}\pi(\frac{h}{2})^3$$
$$V = \frac{\pi}{3*2}h^3$$

Again, differentiating:

$$\frac{dV}{dh} = \frac{\pi}{3*2}3h^2$$
$$\frac{dV}{dh} = \frac{\pi}{2}h^2 \quad : \quad 0 \leq h < 2R$$

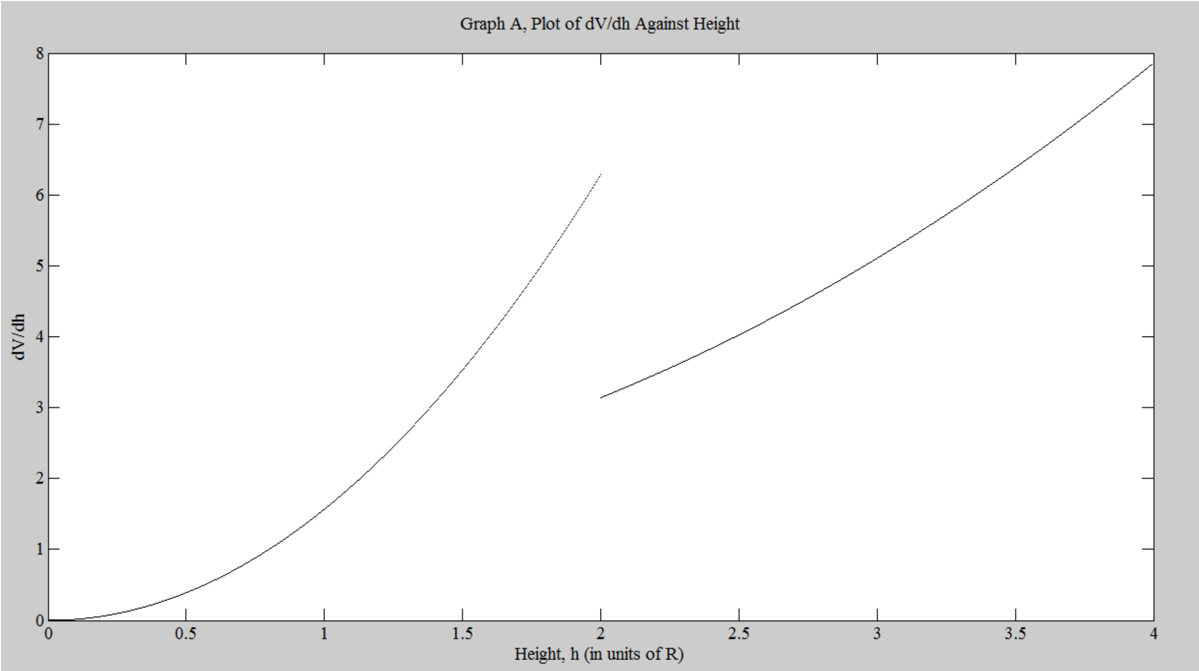
In summary:

$$V(h) = \begin{cases} \frac{\pi}{6}h^3 & \text{if 0 ≤ h < 2R} \\ \\ \frac{\pi}{24}(h^3 + 12hR^2) & \text{if 2R < h ≤ 10R} \end{cases}$$

$$\frac{dV}{dh} = \begin{cases} \frac{\pi}{2}h^2 & \text{if 0 ≤ h < 2R} \\ \\ \frac{\pi}{8}(h^2 + 4R^2) & \text{if 2R < h ≤ 10R} \end{cases}$$

5 Plots of $\frac{dV}{dh}$ Against h and V against h

In this first plot of $\dfrac{dV}{dh}$, the jump discontinuity is readily apparent:



In the second plot, it can be seen that the function V(h) isn't smooth, but it is continuous. As such it therefore belongs to the differentiability class C^0: