Mass on a Hoop

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March 2012

1 Introduction

A particle of mass m is attached to a light circular rigid hoop of radius a which is free to roll in a vertical plane on a rough horizontal table. Initially the hoop stands with the particle at the highest point. It is then displaced slightly. Show that while the hoop is rolling on the table, the speed v of the particle when the radius to the particle makes an angle θ with the upward vertical is given by:

$$v = 2\sqrt{ag}\ sin\frac{\theta}{2}$$

Write down expressions in terms of the variable θ for the horizontal displacement, x, of the particle from its initial position, and its height y, above the table. Hence, or otherwise, show that:

$$\dot{\theta} = \sqrt{\frac{g}{a}}\ tan\frac{\theta}{2}$$

And:

$$\ddot{y} = -2gsin^2\frac{\theta}{2}$$

By considering the reaction of the table on the hoop, or otherwise, describe what happens to prevent the hoop rolling beyond the position for which $\theta = \frac{\pi}{2}$.

2 Freefall

At the point when $\theta = \frac{\pi}{2}$, the mass has no horizontal component to its velocity. At the same time, it's weight will be unsupported by the hoop. Since this is a light hoop, the force on the table due to the weight of the hoop will be zero, and so will the reaction force of the table on the hoop. Therefore the frictional force between the hoop and the table will be zero.

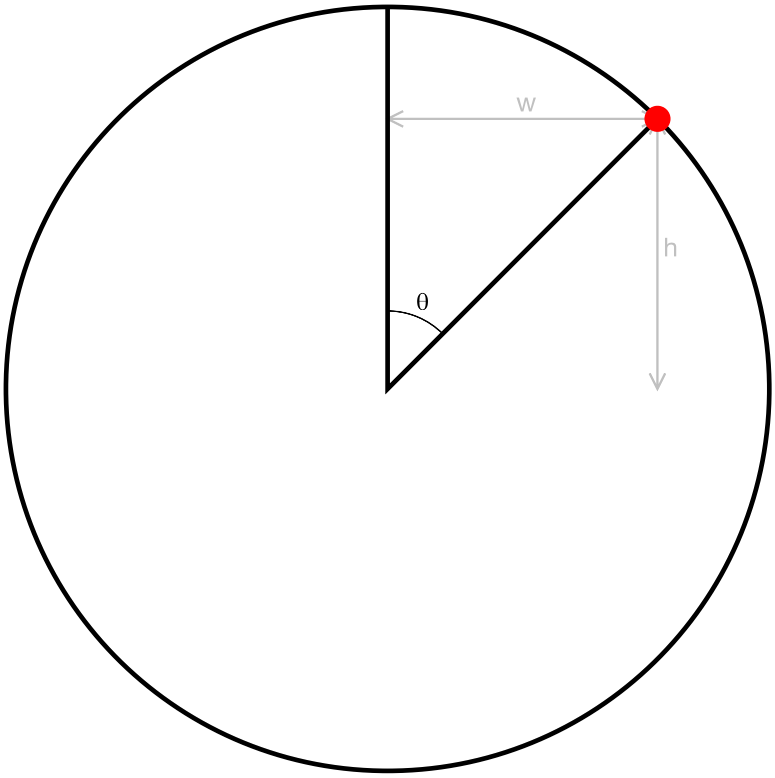
With no frictional force acting on the hoop, it will slip on the table instead of rolling along it. With no force to resist its motion, the mass on the hoop will accelerate vertically downwards at g.

The hoop will actually begin to slip before $\theta = \frac{\pi}{2}$, however, the larger the value of the coefficient of friction between the hoop and the table, the larger angle will be attained before slipping, up to $\frac{\pi}{2}$. Mathematically:

$$\lim_{\mu\to\infty} \theta = \frac{\pi}{2}$$

3 Diagram

Below is a diagram representing the ball some time after it has been displaced.



Expressions for w and h in terms of θ are:

$$w = asin\theta$$
$$h = acos\theta$$

It can be seen that the y position of the mass is given by:

$$y = a + h$$
$$y = a + acos\theta$$
$$y = a(1 + cos\theta)$$

However for the x position, care must be taken to include the arc length in the sum for the horizontal displacement:

$$x = \theta a + w$$
$$x = \theta a + asin\theta$$
$$x = a(\theta + sin\theta)$$

$$x = a(\theta + sin\theta) \tag{1}$$

$$y = a(1 + cos\theta) \tag{2}$$

4 Determining Speed

Since the mass is initially at rest at the top of the loop, all of its energy is in the form of gravitational potential energy. This means that at any later time, it's kinetic and gravitational potential energy must sum to this same amount. Knowing this, an expression for v can be derived:

$$\begin{aligned} GPE_0 &= KE + GPE \\ mg(2a) &= \frac{1}{2}mv^2 + mgy \\ 2ag &= \frac{1}{2}v^2 + gy \\ 4ag &= v^2 + 2gy \\ v^2 &= 4ag - 2gy \end{aligned}$$

Substituting in equation 2:

$$\begin{aligned} v^2 &= 4ag - 2ag(1 + cos\theta) \\ v^2 &= 4ag - 2ag - 2agcos\theta \\ v^2 &= 2ag - 2agcos\theta \\ v^2 &= 2ag(1 - cos\theta) \end{aligned} \tag{3}$$

Now substituting in the following double angle formula:

$$\begin{aligned} cos\theta &= 1 - sin^2\frac{\theta}{2} \\ v^2 &= 2ag(1 - 1 + 2sin^2\frac{\theta}{2}) \\ v^2 &= 2ag(2sin^2\frac{\theta}{2}) \\ v^2 &= 4agsin^2\frac{\theta}{2} \\ v &= 2\sqrt{ag}\ sin\frac{\theta}{2} \end{aligned}$$

Therefore the first part of the problem has been satisfied.

5 Determining Angular Velocity

From sections 3 and 4:

$$x = a(\theta + sin\theta) \tag{1}$$

$$y = a(1 + cos\theta) \tag{2}$$

$$v^2 = 2ag(1 - cos\theta) \tag{3}$$

Differentiating x and y with respect to time will give expressions for the horizontal and vertical velocities:

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt}$$

$$\frac{dx}{d\theta} = a(1 + cos\theta)$$
$$\frac{d\theta}{dt} = \dot{\theta}$$

$$\therefore \dot{x} = a\dot{\theta}(1 + cos\theta) \tag{4}$$

Similarly for y:

$$\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt}$$

$$\frac{dy}{d\theta} = -asin\theta$$
$$\frac{d\theta}{dt} = \dot{\theta}$$

$$\therefore \dot{y} = -a\dot{\theta}sin\theta \tag{5}$$

Since the velocity of the mass is the vector sum of $\dot{x}$ and $\dot{y}$, we can use them like so:

$$\begin{aligned}
v^2 &= \dot{x}^2 + \dot{y}^2 \\
2ag(1 - cos\theta) &= (a\dot{\theta}(1 + cos\theta))^2 + (-a\dot{\theta}sin\theta)^2 \\
2ag(1 - cos\theta) &= a^2\dot{\theta}^2(1 + cos\theta)^2 + a^2\dot{\theta}^2sin^2\theta \\
2g(1 - cos\theta) &= a\dot{\theta}^2((1 + cos\theta)^2 + sin^2\theta) \\
2g(1 - cos\theta) &= a\dot{\theta}^2(1 + 2cos\theta + cos^2\theta + sin^2\theta) \\
2g(1 - cos\theta) &= a\dot{\theta}^2(1 + 2cos\theta + 1) \\
2g(1 - cos\theta) &= a\dot{\theta}^2(2 + 2cos\theta) \\
g(1 - cos\theta) &= a\dot{\theta}^2(1 + cos\theta) \\
\dot{\theta}^2 &= \frac{g}{a}\frac{1 - cos\theta}{1 + cos\theta}
\end{aligned}$$

Now using the following trig identity:

$$tan\frac{\theta}{2} = \sqrt{\frac{1 - cos\theta}{1 + cos\theta}}$$

$$\dot{\theta}^2 = \frac{g}{a}\ tan^2\frac{\theta}{2}$$
$$\dot{\theta} = \sqrt{\frac{g}{a}}\ tan\frac{\theta}{2} \tag{6}$$

Therefore the second part of the problem has been satisfied.

6 Determining Vertical Acceleration

In this section, the following formulae from the previous section will be utilised:

$$\dot{y} = -a\dot{\theta} sin\theta \tag{5}$$

$$\dot{\theta} = \sqrt{\frac{g}{a}}\ tan\frac{\theta}{2} = \sqrt{\frac{g}{a}}\ \frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}} \tag{6}$$

As well as the following trig relationship:

$$sin\theta = 2sin\frac{\theta}{2}\ cos\frac{\theta}{2} \tag{7}$$

The following differential is also used:

$$\frac{d\ sin^2\frac{\theta}{2}}{d\theta} = \frac{sin\theta}{2} \tag{8}$$

Combining equations 5 and 6 and using relationship 7:

$$\begin{aligned}
\dot{y} &= -a\dot{\theta} sin\theta \\
\dot{y} &= -2a\dot{\theta} sin\frac{\theta}{2}\ cos\frac{\theta}{2} \\
\dot{y} &= -2a\sqrt{\frac{g}{a}}\ sin\frac{\theta}{2}\ cos\frac{\theta}{2}\ \frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}} \\
\dot{y} &= -2\sqrt{ag}\ sin^2\frac{\theta}{2}
\end{aligned}$$

Now differentiating:

$$\frac{d\dot{y}}{dt} = \frac{d\dot{y}}{d\theta}\frac{d\theta}{dt}$$

$$\begin{aligned}
\frac{d\dot{y}}{d\theta} &= -2\sqrt{ag}\ \frac{sin\theta}{2} = -\sqrt{ag}\ sin\theta \\
\frac{d\theta}{dt} &= \dot{\theta}
\end{aligned}$$

$$\begin{aligned}
\ddot{y} &= -\sqrt{ag}\ \dot{\theta} sin\theta \\
\ddot{y} &= -2\sqrt{ag}\ \dot{\theta} sin\frac{\theta}{2}\ cos\frac{\theta}{2} \\
\ddot{y} &= -2\sqrt{ag}\ \sqrt{\frac{g}{a}}\ sin\frac{\theta}{2}\ cos\frac{\theta}{2}\ \frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}} \\
\ddot{y} &= -2g\ sin^2\frac{\theta}{2}
\end{aligned}$$

Therefore the third part of the problem has been satisfied.